Efficient Coherent Control by Sequences of Pulses of Finite Duration

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Reliable long-time storage of arbitrary quantum states is a key element for quantum information processing. In order to dynamically decouple a spin or quantum bit from a dephasing environment by non-instantaneous pulses, we introduce an optimized sequence of N control π pulses which are realistic in the sense that they have a finite duration and a finite amplitude. We show that optimized dynamical decoupling is still applicable and that higher-order decoupling can be reached if shaped pulses are implemented. The sequence suppresses decoherence up to the order $\mathcal{O}(T^{N+1}) + \mathcal{O}(\tau_{\mathrm{mx}}^M)$, with T the total duration of the sequence and τ_{mx} the maximum length of the pulses. The exponent $M \in \mathbb{N}$ depends on the shape of the pulse. Based on existing experiments, a concrete setup for the verification of the properties of the advocated sequence is proposed.

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I. INTRODUCTION

In quantum information processing (QIP) and in nuclear magnetic resonance (NMR) it is essential to be able to decouple the quantum bit or the spin, respectively, from its environment. Both fields of research are of widespread interest and relevance. In the former the ultimate goal is to realize reliable long-time storage of quantum information with as low as possible error rates. This is a prerequisite for QIP[1,2]. In the latter, the high-precision measurement of nuclear spin dynamics is a long-standing goal[3,4].

Besides choosing well-isolated systems the application of appropriately tailored sequences of control pulses[3,5,6], i.e., dynamic decoupling (DD), is one of the promising routes to this goal. The basic idea goes back to Hahn's spin echo pulse which averages a static perturbance to zero[7]. For a dynamic environment, or bath, sequences of pulses are required[5,6,8–10]. The early suggestions are essentially periodic in time.

Recently, the additional advantages of sequences with non-equidistant pulses were discovered. Concatenation (CDD) can suppress unwanted couplings in a high power T^l of the length of the sequence[11]. But for the method used in Ref. 11, the required number N of pulses grows exponentially with 4^ℓ. For pure dephasing, it was shown that this growth can be reduced decisively to a linear one $l \propto N$ if the instants of π pulses were chosen according to

$$t_j = T \sin^2(j\pi/(2N+2)), \tag{1}$$

which is called UDD (Uhrig DD). The relation (1) was derived for a spin-boson model[12] where it was observed that no details of the model entered. On the basis of numerical evidence and finite order recursion it was conjectured that (1) is applicable to any dephasing model[13,14]. This claim was finally proven[15] for any order in T. For various simulated classical noise spectra the experimental verification of the theoretical results was achieved[16–18] by microwave control of the transition in Be ions. The suppression of the decoherence of the electron spin of hydrogen radicals was investigated by electron spin resonance in crystals of irradiated malonic acid[19]. The decoherence was due to the quantum noise induced by nuclear spins. Again, the UDD proved superior to standard sequences.

For general decoherence, concatenation of the UDD sequence (CUDD) can be used[20]. For a suppression of the decoherence up to T^ℓ, the number of pulses grows as 2^ℓ which is an improvement by a square root with respect to the CDD of Ref. 11. A more efficient scheme, called quadratic DD (QDD), which requires only a quadratically growing number of pulses, has been proposed very recently based on numerical[21] and analytical evidence[22].

All these sequences (periodic DD[10], CDD[11], UDD[12], CUDD[20], and QDD[21,22]) rely on instantaneous, thus idealized, pulses. This problem was realized early on and ongoing research investigates pulses of finite duration τ_{p}[23–27] and sequences of such pulses[28–32]. Eulerian DD[28,33] is designed to annihilate the first order of a Magnus expansion over the whole sequence. Thus corrections of the order of $T\tau_{\mathrm{p}}$ are not excluded. Similar caveats apply to many other sequences[29–32]. Also the experimental realizations in Refs. 16–18 have to take into account that real pulses cannot be instantaneous because the control amplitudes are necessarily bounded.

Our aim here is to derive an optimized sequence with UDD properties which relies on realistic pulses of finite duration and which is adapted to these real pulses. We do not provide a general scheme to use pulses of bounded control for arbitrary DD sequences. If the shape is appropriately designed, the pulse can be approximated as an instantaneous one up to $\mathcal{O}(\tau_{\mathrm{p}}^M)$. For $M = 3$ explicit results were derived in Ref. 34 while a recursive scheme for arbitrary M has been proposed recently[35]. As far as the correction $\mathcal{O}(\tau_{\mathrm{p}}^M)$ is negligible, the proposed sequence displays the same exact analytic properties as the UDD sequence of ideal, instantaneous pulses.

We approach the problem hierarchically. That means that we use the pulses and the periods of free evolution as building blocks for the sequence. First, the properties

of the pulses are derived and discussed. Second, these properties are used in the sequence. To this end, we exploit scaling in two independent variables, namely the durations $\tau_{\rm p}$ of the pulses, whose maximum is $\tau_{\rm mx}$, and the total duration of the sequence denoted by T. Note that these two time scales are largely independent, both in theory and in experiment, because the pulses are not applied back to back. The only constraint is that T must be larger than the sum of the pulse durations $\tau_{{\rm p},j}$ (belonging to pulse j)

$$T \geq \sum_{j=1}^{N} \tau_{{\rm p},j}. \tag{2}$$

Relying on pulses, which cancel all orders $m < M$, the whole sequence avoids all mixed terms $T^n \tau_{\rm mx}^m$, where $n \leq N+1$ and $m < M$. Hence important progress over existing proposals[28–32] is achieved.

II. MODEL

We start from the Hamiltonian

$$H = \hat{A}_0 + \sigma_z \hat{A}_1 \widetilde{F}(t) \tag{3}$$

where σ_z is the z component of the Pauli matrices. It is acting on the $S = 1/2$ spin or, generally, on the two-level system which represents a qubit. The operators $\hat{A}_i$ act on the bath only; they may also be c-numbers. We consider any kind of bath with bounded operators for the sake of the mathematical argument $||H|| \leq \gamma < \infty$, where $||\cdot||$ is any appropriate operator norm which remains invariant under unitary transforms. We expect that the order of suppression of the decoherence holds for any bath which can be approximated by bounded baths, i.e., the bath should have a hard high-energy cutoff.

No spin flip terms are included in (3) implying an infinite spin-lattice relaxation time T_1. It is an excellent approximation if $T_1 \gg T_2$ where T_2 is the dephasing time. Such a situation is achieved in the rotating reference frame of a system where the two levels with eigenvalues ± 1 of σ_z lie energetically far apart. Longitudinal relaxation and general decoherence will be addressed below.

Moreover, (3) is the effective Hamiltonian in the interaction picture of the short control pulses[12–15]. Thus the switching function $\widetilde{F}(t) \in \mathbb{R}$ appears. The simplest example is an instantaneous π pulse at $t = t_j$ which realizes a rotation about an axis perpendicular to σ_z. Then $\widetilde{F}(t)$ changes its sign at $t = t_j$ abruptly while it is constant elsewhere. A sequence of such pulses at the instants $\{t_j\}$ with $j \in \{1, 2, \ldots, N\}$ implies $\widetilde{F}(t) = (-1)^j$ for $t \in \{t_j, t_{j+1}\}$ where we define $t_0 = 0$ and $t_{N+1} = T$.

Our derivation is based on the bounded quantum model (3). Thereby, classical Gaussian noise is treated to the extent that it can be approximated by the quantum model[14]. Certainly, non-Gaussian classical noise, see for instance Ref. 36, should be considered separately which is beyond the scope of the present article.

III. DERIVATION

Optimization of the sequence means to ask the question which switching instants t_j make the sequence $\{t_j\}$ most efficient. For ideal instantaneous pulses, it was shown that the UDD instants (1) are optimum in the sense that the time evolution depends on the spin weakly[12–15]. The time evolution operator

$$\hat{U}_\pm = \prod_{j=0}^{N} e^{-i[\hat{A}_0 \pm \hat{A}_1 \tilde{F}(t_j)](t_{j+1}-t_j)} \tag{4}$$

for the eigenstates of σ_z with eigenvalues ± 1 depends on the spin only in a high power of T

$$\hat{U}_+ - \hat{U}_- = \mathcal{O}((\gamma T)^{N+1}). \tag{5}$$

The analytical derivation of (5) is achieved by direct time-dependent perturbation theory (TDPT)[15] in powers of tH. Thus, if the $N+1$st power does not vanish it is of order $(\gamma T)^{N+1}$. The iterated time integrations of TDPT are conveniently expressed by the substitution $t := T \sin^2(\vartheta/2)$ as integrations over the variable ϑ. The instants (1) are equidistant if expressed in ϑ because for $F(\vartheta) := \widetilde{F}(T \sin^2(\vartheta/2))$ we have

$$F_{\rm UDD}(\vartheta) = (-1)^j \quad \text{for} \quad \vartheta \in \left(\frac{j\pi}{N+1}, \frac{(j+1)\pi}{N+1}\right) \tag{6}$$

with $j \in \{0, N\}$. Allowing j to take all integer values $j \in \mathbb{Z}$ the function $F_{\rm UDD}$ becomes an odd function with $F_{\rm UDD}(\vartheta + \pi/(N+1)) = -F_{\rm UDD}(\vartheta)$. Hence the Fourier series of $F_{\rm UDD}(\vartheta)$ comprises only odd sin harmonics $\sin(l(N+1)\vartheta)$ with $l \in \{1, 3, \ldots\}$. The coefficients are $4/(\pi l)$. From this property, (5) is derived by exploiting trigonometric addition theorems recursively[15].

The power of the UDD sequence has been demonstrated experimentally[16,17]. The noise, i.e., the coupling to the bath $\hat{A}_1$, is simulated, so that it can be switched off during the pulse. Thereby, a partial solution of the finiteness of the pulse amplitudes is achieved. But generally decoherence processes cannot be switched off. Using pulses of duration $\tau_{\rm p}$ with constant amplitude instead of instantaneous pulses introduces an unwanted term of the order $\gamma\tau_{\rm p}$ at each rotation, i.e., linear in the pulse length. For a sequence of length N these corrections can accumulate to $N\gamma\tau_{\rm p}$ unless the contributions of subsequent pulses cancel each other.

An improvement by one order in $\gamma\tau_{\rm p}$ is achieved by the ersatz π pulse which makes the linear correction vanish. Then the time evolution operator $\hat{U}_{\rm p}$ for a pulse reads

$$\hat{U}_{\rm p}(t+\tau_{\rm p}, t) = \hat{U}_{\rm p}^{\rm ideal}(t+\tau_{\rm p}, t) + \mathcal{O}((\gamma\tau_{\rm p})^M) \tag{7a}$$

$$\hat{U}_{\rm p}^{\rm ideal}(t+\tau_{\rm p}, t) = e^{-{\rm i}(\tau_{\rm p}-\tau_{\rm s})H} \hat{P}_\theta e^{-{\rm i}\tau_{\rm s}H}, \tag{7b}$$

where $M = 2$. It is understood that t marks the beginning of the pulse and $t+\tau_\mathrm{p}$ its end. It is important that H is the Hamiltonian of the total system, i.e., spin, bath, and their mutual coupling, $\hat{P}_\theta$ is the ideal pulse with $\theta = \pi$, and τ_s is the instant when the approximated ideal pulse occurs. In a sequence $\{t_j\}$, the instant τ_s is to be identified with the switching instants t_j. No adjustment of the sequence takes place. Relation (7) can be achieved by shaping the pulse appropriately[25–27]. Hence we can set up a UDD sequence with more realistic pulses of the kind (7) for which the deviations read

$$\hat{U}_+^{\mathrm{UDD}} - \hat{U}_-^{\mathrm{UDD}} = \mathcal{O}\left((\gamma T)^{N+1}\right) + \mathcal{O}\left(N(\gamma\tau_{\mathrm{mx}})^M\right) \quad (8)$$

with $M = 2$. The additivity of the corrections is a straightforward property of the unitary evolution operators. If we denote the UDD sequence made from the ideal pulses $\hat{U}_\mathrm{p}^{\mathrm{ideal}}(t+\tau_\mathrm{p}, t)$ in (7b) by $\hat{U}_\pm^{\mathrm{UDD,ideal}}$, we know from (7a)

$$\hat{U}_\pm^{\mathrm{UDD}} = \hat{U}_\pm^{\mathrm{UDD,ideal}} + \mathcal{O}(N(\gamma\tau_{\mathrm{mx}})^M) \quad (9)$$

for N pulses. The unitary invariance of the norm γ is used for each pulse. Next, we know from the properties of the UDD sequence[13–15]

$$\hat{U}_+^{\mathrm{UDD,ideal}} - \hat{U}_-^{\mathrm{UDD,ideal}} = \mathcal{O}((\gamma T)^{N+1}). \quad (10)$$

Combined with (9) this equation implies (8).

The bound $(\gamma T)^{N+1}$ resulting from the sequence can be improved systematically by enlarging N. The bound $N(\gamma\tau_{\mathrm{mx}})^2$ resulting from the pulses can be improved by making it shorter. But if this is not possible, one is stuck because the exponent of 2 cannot be incremented for $\theta = \pi$ pulses as implied by mathematical no-go theorems[25,26]. Hence we are facing here a serious conceptual obstacle.

Recently a variant of (7)

$$\begin{aligned} \hat{U}_\mathrm{p}(t+\tau_\mathrm{p}, t) &= \hat{U}_\mathrm{p}^{\mathrm{zero}}(t+\tau_\mathrm{p}, t) + \mathcal{O}((\gamma\tau_\mathrm{p})^M) \quad (11a)\\ \hat{U}_\mathrm{p}^{\mathrm{zero}}(t+\tau_\mathrm{p}, t) &= e^{-\mathrm{i}\tau_\mathrm{p}\hat{A}_0}\hat{P}_\theta \quad (11b)\end{aligned}$$

with $M = 3$ was shown[34] to reduce the correction to $(\gamma\tau_\mathrm{p})^3$. Note that in (11) only the Hamiltonian $\hat{A}_0$ of the bath occurs without coupling to the spin. Hence $[\hat{A}_0, \hat{P}_\theta] = 0$ holds and no τ_s needs to be introduced.

Explicit solutions are obtained for pure dephasing[34]. The correlation time of the dephasing bath should not be much smaller than τ_p. Moreover, no no-go theorem was found which prevents to achieve higher orders as well. Indeed, a recursive scheme based on concatenation is proposed which achieves arbitrary order M at exponential cost[35], i.e., each composite π pulse consists of $> 17^{M-1}$ elementary pulses. This demonstrates that in principle arbitrary M can be achieved though the exponential cost may spoil its practical usefulness. But due to the shortness of the pulses compared to the whole sequence ($\tau_\mathrm{p} \ll T$) we do not expect that particularly large values of M are required.

The property (11) is promising, but it *cannot* be used in standard DD, or in UDD in particular, as ersatz for an instantaneous pulse. This is so because any standard DD sequence presupposes that between the pulses $\hat{P}_\pi$ the full Hamiltonian H, not only $\hat{A}_0$, is active. This conceptual obstacle cannot be solved by pulse shaping because the no-go theorems block further progress[25,26]. To overcome this obstacle is the main achievement of the present paper. We find that an adjustment of the sequence to the pulses of finite duration is required.

Our present fundamental observation is that relation (11b) translates to $\widetilde{F}(t)$ for a single realistic pulse between t^- and $t^+ = t^- + \tau_\mathrm{p}$ in the form

$$\widetilde{F}(t) = \begin{cases} 1 & \text{for} \quad t < t^- \\ 0 & \text{for} \quad t^- < t < t^+ \\ -1 & \text{for} \quad t > t^+ \end{cases}. \quad (12)$$

The π pulse implies the inversion of the sign. But during the pulse itself the relation (11) implies that the coupling between spin (qubit) and bath is effectively averaged to zero up to $\mathcal{O}((\gamma\tau_\mathrm{p})^M)$. This is so since $\hat{A}_0$ in (11b) does not comprise the spin-bath coupling; it only comprises the bath dynamics. This implies that the switching function $\widetilde{F}(t)$ takes the value zero during the pulse. Note that there are jumps in the switching function even though the pulse is generated by bounded control. The reason for this behavior is that unitary time evolution is considered over *finite* time intervals, not over infinitesimal intervals. This means that from the hierarchical level of the sequence we do not look into the pulses. The description with $\widetilde{F}(t)$ is only valid on the level of the sequence, not within the pulse interval. Furthermore, the correction term in Eq. (11a) may not be forgotten.

Next we look for a sequence with $\widetilde{F}(t) \in \{-1, 0, 1\}$ which corresponds to an odd function $F(\vartheta) \in \{-1, 0, 1\}$ with the antiperiodic behavior $F(\vartheta + \pi/(N+1)) = -F(\vartheta)$. Such a sequence, RUDD (realistic UDD), allows for the same mathematical argument as UDD (6) ensuring that the effective time evolution of the spin is the identity up to corrections of the order $(\gamma T)^{N+1}$. The reason is the antiperiodicity of the switching function which is the fundamental reason for the annihilation of the preceding orders[15,22]. Note that this argument holds only for UDD and similar optimized sequences. Hence we do not provide a general scheme for the incorporation of pulses of finite duration into arbitrary sequences.

The sequence fulfilling the requirement of antiperiodicity reads

$$F_{\mathrm{RUDD}}(\vartheta) = \begin{cases} (-1)^j & \text{for } \vartheta \in \left(\frac{j\pi}{N+1} + \vartheta_\mathrm{p}, \frac{(j+1)\pi}{N+1} - \vartheta_\mathrm{p}\right) \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

for $j \in \mathbb{Z}$. The Fourier series comprises only the odd sin harmonics $\sin(l(N+1)\vartheta)$ with coefficients $4\cos(l(N+1)\vartheta_\mathrm{p})/(\pi l)$. The parameter $0 \leq \vartheta_\mathrm{p} \leq \pi/(2N+2)$ determines the duration of the pulses. Except for the given inequality it is independent of N. Note that

pulses of equal duration in ϑ do not correspond to pulses of equal duration in time t

$$\widetilde{F}_{\mathrm{RUDD}}(t) = \begin{cases} (-1)^j & \text{for } t \in \left(t_j^+, t_{j+1}^-\right) \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

with $t_j^\pm := T\sin^2\left[\frac{j\pi}{2N+2} \pm \vartheta_{\mathrm{p}}/2\right]$. This is illustrated for $N = \{1, 2, 3, 4\}$ in Fig. 1 where also the necessary time-dependent amplitudes $v(t)$ defining the control Hamiltonian $H_{\mathrm{C}}(t) = v(t)\sigma_y$ are shown; σ_y is the y component of the Pauli matrices. For instances, the amplitudes can be parametrized by

$$\begin{aligned} v_\theta(t) = \theta/2 + (a_\theta - \theta/2)\cos(2\pi t/\tau_{\mathrm{p}}) + \\ (b_\theta - a_\theta)\cos(4\pi t/\tau_{\mathrm{p}}) + \\ (c_\theta - b_\theta)\cos(6\pi t/\tau_{\mathrm{p}}) - c_\theta\cos(8\pi t/\tau_{\mathrm{p}}). \end{aligned} \tag{15}$$

There is one subtlety about the beginning and the end of the sequence. In order to generate the switching function (14) there must be a first and a last pulse which averages the coupling between spin and bath to zero while inducing *no* net rotation. For this purpose θ can take any multiple of 2π. Solving the equations derived in Ref. 34, which imply that the pulse fulfills the relation (11), leads to the parameters given in the caption of Fig. 1.

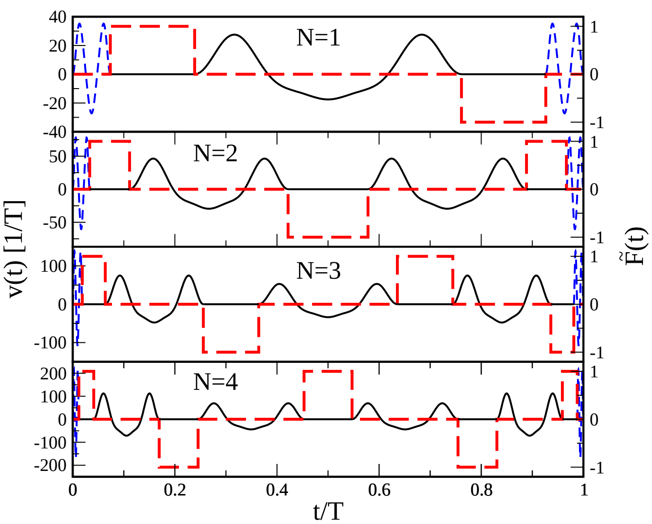

FIG. 1: Amplitudes $v(t)$ (black solid and blue short-dashed lines) for pulses rotating about axes in the xy plane and the resulting switching functions $\widetilde{F}(t)$ (red dashed lines) for $N = 1, 2, 3, 4$ π pulses. For clarity, fairly large values $\vartheta_{\mathrm{p}} = 0.7\pi/(2N+2)$ are chosen for the sake of clarity. For rendering purposes, ϑ_{p} is chosen here to depend on N. Generally, it only has to fulfil $\vartheta_{\mathrm{p}} \leq \pi/(2N+2)$. Black solid lines stand for $\theta = \pi$ pulses with $v_\pi(t)$ as in (15) with $a_\pi = 10.804433[1/\tau_{\mathrm{p}}]$, $b_\pi = 6.831344[1/\tau_{\mathrm{p}}]$, $c_\pi = 2.174538[1/\tau_{\mathrm{p}}]$. The first and the last pulse rotates by 2π (blue short-dashed lines) with $a_{2\pi} = 10.236155[1/\tau_{\mathrm{p}}]$, $b_{2\pi} = 2.9661717[1/\tau_{\mathrm{p}}]$, $c_{2\pi} = 0.889052[1/\tau_{\mathrm{p}}]$. For clarity, $v_{2\pi}(t)/5$ is plotted.

The duration of the pulses is shortest towards the ends of the interval T for which the quantum state of the spin is to be stored. Concomitantly, the amplitudes are largest for the first and the last pulse. In practice, the initial and the final pulse can be combined with the pulses by which the quantum state of the spin (the qubit) is generated, for instance a $\pi/2$ pulse (for solutions see Ref. 34).

What has been achieved by the sequence (14) depicted in Fig. 1? This sequence is an optimized dynamic decoupling scheme made from pulses of finite duration and finite amplitudes with analytically founded properties. Bounded control is a crucial aspect for realistic sequences, so that the proposed sequence is an important step closer to a realistic scenario. Nevertheless, the sequence is still optimized in the sense that it shares the same power law property as the UDD built from instantaneous pulses.

There are two sources for corrections in the unitary time evolution $\hat{U}_\pm^{\mathrm{RUDD}}$ of the RUDD sequence. The first kind of corrections stems from the pulses which are only close to $\hat{U}_{\mathrm{p}}^{\mathrm{zero}}(t+\tau_{\mathrm{p}}, t)$ but not identical to it, see Eq. (11). Denoting the time evolution of the RUDD sequence made from the pulses $\hat{U}_{\mathrm{p}}^{\mathrm{zero}}(t+\tau_{\mathrm{p}}, t)$ in (11b) by $\hat{U}_\pm^{\mathrm{RUDD,zero}}$ we know from (11a)

$$\hat{U}_\pm^{\mathrm{RUDD}} = \hat{U}_\pm^{\mathrm{RUDD,zero}} + \mathcal{O}(N(\gamma\tau_{\mathrm{mx}})^M) \tag{16}$$

for N pulses. The second kind of corrections stems from the sequence itself. The time evolution $\hat{U}_\pm^{\mathrm{RUDD,zero}}$ is rigorously governed by $F_{\mathrm{RUDD}}(\vartheta)$ defined in (13). Then we know from Ref. 15 that

$$\hat{U}_+^{\mathrm{RUDD,zero}} - \hat{U}_-^{\mathrm{RUDD,zero}} = \mathcal{O}((\gamma T)^{N+1}). \tag{17}$$

The total correction is given by the sum of both kinds of corrections because their norm is invariant under unitary transformations. So in analogy to (8) we obtain

$$\hat{U}_+^{\mathrm{RUDD}} - \hat{U}_-^{\mathrm{RUDD}} = \mathcal{O}\left((\gamma T)^{N+1}\right) + \mathcal{O}\left(N(\gamma\tau_{\mathrm{mx}})^M\right). \tag{18}$$

We stress that this relation excludes mixed terms $(\gamma T)^n(\gamma\tau_{\mathrm{mx}})^m$ with $n \leq N+1$ and $m < M$ because each pulse complies with (11a) separately. We point out that M does not need to be as large as N because the pulses are much shorter anyway. So the relatively short and simple pulse found in Ref. 34 realizing $M = 3$ may often be completely sufficient.

For later reference, we point out that the above derivation also holds if we allow for an explicit analytic time dependence of the operators $\hat{A}_0$ and $\hat{A}_1$ in (3). This was recently shown by us in the context of optimized dynamic decoupling for time dependent Hamiltonians[22] relying only on the mathematical properties of the switching function $F(\vartheta)$. Hence the same argument also applies to the RUDD sequence if the pulses are shaped to realize zero coupling during their duration, see Eq. (12). This is definitely the case if there is no time dependence *during* the pulses because the pulses suggested in Refs. 34 and 35 can be used. This is indeed a relevant case as we will discuss below.

We emphasize that the RUDD with (18) provides an efficient scheme for dynamic decoupling based on bounded

control. It is the main result of our paper. The previously obstructive no-go theorems[25,26] can be circumvented by the RUDD approach. The qualitative novel finding in the present work is that the sequence has to be adjusted in a precise way in order to allow for realistic pulses while preserving the properties of the sequence of ideal pulses. Above we constructed a precise prescription which achieves the necessary adjustment. We expect that this observation extends beyond the case of UDD and RUDD. This expectation is illustrated in the next section.

We emphasize that the number of pulses N cannot be made infinite without using shorter and shorter pulses with larger and larger amplitudes. Hence a given bound to the available power of the control pulses limits the maximum possible number of pulses for a given interval T. But such limits exist in any experimental setup anyway[16,17,19] and we expect that the RUDD approach will prove its usefulness for a moderate number of pulses. The limit $T \to 0$ is studied here to characterize the mathematical properties of the idealized situation. The achievement of the RUDD over the UDD sequence is that for any finite duration T and finite number of pulses N only pulses of finite amplitude are needed.

IV. ITERATED SEQUENCES

In view of the above we expect that the famous CPMG sequence[8,9] can be improved for realistic pulses as follows. The CPMG is given by the n-fold iteration of the two-pulse cycle $t - \pi - 2t - \pi - t$, where π stands for a π pulse and t for free evolution of time t. This two-pulse cycle is the UDD sequence for $N = 2$ pulses[12]. Hence for pulses of finite duration the iteration of the $N = 2$ panel in Fig. 1 suggests itself. A slight modification is possible by replacing two 2π pulses, where two cycles meet, by one 2π pulse of double the length. Hence it is promising to use the sequence

$$\begin{aligned}(2\pi)_{t_1}\left[-t_2-\pi_{\tau_\mathrm{p}}-2t_2-\pi_{\tau_\mathrm{p}}-t_2-(2\pi)_{2t_1}\right]^{n-1}-\\ -t_2-\pi_{\tau_\mathrm{p}}-2t_2-\pi_{\tau_\mathrm{p}}-t_2-(2\pi)_{t_1},\end{aligned}\tag{19}$$

with $t_1 = 2t(1-\cos(\vartheta_\mathrm{p}))$, $t_2 = 2t\sin((\pi/6)-\vartheta_\mathrm{p})$, and $\tau_\mathrm{p} = 4t\cos(\pi/6)\sin(\vartheta_\mathrm{p})$. The subscripts indicate the pulse durations. We iterate that the advocated recipe to account for bounded control only applies to UDD-type sequences.

V. SIMULATION OF A RUDD SEQUENCE

The advocated RUDD sequence relies on its mathematical properties which have a certain beauty in themselves. But the ultimate check will be its experimental usefulness. A crucial step on this route is an experiment with simulated noise such as the one performed for UDD[16–18]. There, the simulated noise was switched off during the pulse. The theoretical calculations took this dead time of the noise into account. But variable pulse lengths such as in RUDD were not considered.

We propose to implement the RUDD according to (14) with pulses of finite, constant amplitudes during the intervals where $\widetilde{F}(t) = 0$. No pulse shaping is required if the noise is switched off during the pulse so that $\widetilde{F}(t) = 0$ is fulfilled by construction. Hence we have

$$\hat{U}_+^{\mathrm{RUDD}} - \hat{U}_-^{\mathrm{RUDD}} = \mathcal{O}\left((\gamma T)^{N+1}\right)\tag{20}$$

for this particular experiment instead of (18).

The pulse intervals have to be chosen as in (14). Concomitantly the amplitudes have to vary to ensure that the pulses are π pulses. In this way any deviation resulting from the pulses is eliminated. It is highly interesting to investigate if such a RUDD sequence is more powerful than existing realizations.

VI. LONGITUDINAL RELAXATION

A UDD sequence can also suppress longitudinal relaxation[15]. Pulses of angle π about the z axis can suppress terms proportional to x and y component, i.e., σ_x and σ_y, of the Pauli matrices up to order $(\gamma T)^{N+1}$ for $\{t_j\}$ as in (1). Concatenation of such UDD sequences (CUDD) can be used to suppress any kind of relaxation[20]. The QDD appears to be the most efficient scheme to fulfill this purpose[21,22].

The pulses depicted in Fig. 1 and computed in Ref. 34 also work to order $(\gamma\tau_\mathrm{p})^3$ if used for rotations $\hat{P}_\theta^z$ around the z axis for arbitrary couplings to σ_x, σ_y, and σ_z. The pulse $\hat{P}_\pi^z$ induces an inversion of the sign of the couplings along σ_x and σ_y.

To see this one has to modify the specific calculation for a rotation about a fixed axis in Ref. 34 according to $\hat{A}_0 \to \hat{B}_0 = \hat{A}_0 + \hat{A}_z\sigma_z$ and $\vec{\sigma}\cdot\vec{A} \to \vec{\sigma}_\perp\cdot\vec{A} = \sigma_x\hat{A}_x + \sigma_y\hat{A}_y$ for a Hamiltonian $H = \hat{A}_0 + \vec{\sigma}\cdot\vec{A}$. The Eq. (11) becomes

$$\hat{U}_\mathrm{p}(t+\tau_\mathrm{p}, t) = e^{-\mathrm{i}\tau_\mathrm{p}\hat{B}_0}\hat{P}_\theta^z\hat{U}_G(\tau_\mathrm{p}, 0),\tag{21}$$

where $\hat{U}_G(\tau_\mathrm{p}, 0)$ encodes the corrections. It is given by

$$\hat{U}_G(\tau_\mathrm{p}, 0) = T\left\{e^{-\mathrm{i}\int_0^{\tau_\mathrm{p}} G(t)\mathrm{d}t}\right\}.\tag{22}$$

where the time dependent Hamiltonian of the corrections $G(t)$ stands for

$$G(t) := e^{\mathrm{i}\hat{B}_0 t}(\hat{P}_t^z)^\dagger\left(\vec{\sigma}_\perp\cdot\vec{A}\right)\hat{P}_t^z e^{-\mathrm{i}\hat{B}_0 t}\tag{23}$$

with $\hat{P}_t^z = \exp\left(-i\sigma_z\int_0^t \mathrm{d}s\ v(s)\right)$ resulting from $H_\mathrm{C}(t) = v(t)\sigma_z$ representing the pure control rotation at instant t. We show that $\hat{U}_G(\tau_\mathrm{p}, 0) = 1 + \mathcal{O}((\gamma\tau_\mathrm{p})^3)$ if the shaped rotations about σ_y proposed in Ref. 34 are applied about σ_z.

The Magnus expansion[3,37] allows us to write the time evolution in Eq. (22) in terms of cumulants $U_G(\tau_{\rm p},0) = \exp\left(-i\tau_{\rm p}\sum_{i=1}^{\infty}\eta^{(i)}\right)$. Each cumulant $\eta^{(i)}$ scales as $(\gamma\tau_{\rm p})^i$. Following the approach of Ref. 34, it is straightforward to find

$$\eta^{(1)} = \eta_{11}\left(\sigma_x\hat{A}_y - \sigma_y\hat{A}_x\right) + \eta_{12}\left(\sigma_x\hat{A}_x + \sigma_y\hat{A}_y\right), \quad (24)$$

with η_{11} and η_{12} the first order corrections. For the second order one finds $\eta^{(2)} = \eta^{(2a)} + \eta^{(2b)}$ with

$$\begin{aligned} \eta^{(2a)} &= \eta_{21}[\hat{B}_0, \sigma_x\hat{A}_y - \sigma_y\hat{A}_x] \\ &\quad + \eta_{22}[\hat{B}_0, \sigma_x\hat{A}_x + \sigma_y\hat{A}_y] \qquad (25a) \\ \eta^{(2b)} &= 2\eta_{23}\sigma_z\left\{[\hat{A}_y, \hat{A}_x] + i(\hat{A}_x^2 + \hat{A}_y^2)\right\}. \qquad (25b) \end{aligned}$$

The expressions for η_{21}, η_{22}, and η_{23} are

$$\begin{aligned} \eta_{11} &:= \int_0^{\tau_{\rm p}} {\rm d}t \sin\psi(t) \qquad (26a) \\ \eta_{12} &:= \int_0^{\tau_{\rm p}} {\rm d}t \cos\psi(t) \qquad (26b) \\ \eta_{21} &:= \int_0^{\tau_{\rm p}} {\rm d}t\ t\ \sin\psi(t) \qquad (26c) \\ \eta_{22} &:= \int_0^{\tau_{\rm p}} {\rm d}t\ t\ \cos\psi(t) \qquad (26d) \\ \eta_{23} &:= \iint_0^{\tau_{\rm p}} {\rm d}t_1{\rm d}t_2 \sin(\psi(t_1) - \psi(t_2)){\rm sgn}(t_1 - t_2), \qquad (26e) \end{aligned}$$

where $\psi(t) := 2\int_0^t v(t')dt'$. These conditions are exactly the same as those reported in Ref. 34 for pure dephasing. Hence they have the same solutions and the pulses depicted in Fig. 1 make the first and the second order corrections vanish also for longitudinal relaxation. No changes in the pulse shapes are required. Up to the third order, the transverse coupling is suppressed and only the z-coupling survives unaltered. Between two subsequent pulses the sign of the x and y coupling is inverted. For pulses corrected in higher order $M > 3$ corrections we again refer to Ref. 35 for a proof-of-principle construction. Hence, the RUDD sequence is equally applicable for the suppression of longitudinal relaxation.

Eqs. (26) hold generally for the suppression of decoherence perpendicular to the fixed axis of rotation of the pulse. The decohering coupling along this axis is not suppressed. The case of pure dephasing can be seen as special case of the more general case discussed here: There is no coupling along the axis of rotation and only one (out of two possible) perpendicular coupling.

VII. CONCATENATION OF RUDD SEQUENCES

To tackle general decoherence the combination of at least two sequences of rotations about perpendicular spin axes are used. Available schemes rely on recursive concatenation as for CDD[11] or CUDD[20] or on a single step concatenation as for QDD[21,22]. Hence it is natural to consider concatenation of RUDD sequences of rotations about two perpendicular spin axes.

For simplicity we consider the QDD scheme which comprises two levels. On the first level two (e.g., $\hat{A}_x$ and $\hat{A}_y$) of the three couplings $\vec{A}$ to the components of $\vec{\sigma}$ are eliminated up to a certain order. This is exactly what is achieved by a RUDD of N_z rotations about z for longitudinal relaxation as discussed in the previous section.[38] Up to the corrections $\mathcal{O}\left((\gamma T_z)^{N_z+1}\right) + \mathcal{O}\left(N(\gamma\tau_{z,\rm mx})^M\right)$ of the primary level, the resulting time evolution is given an effective Hamiltonian which implies dephasing only. Note that T_z is the duration of the primary RUDD sequence.

The effective Hamiltonian is of the form given in Eq. (3), but with time dependent operators $\hat{A}_0(t)$ and $\hat{A}_z(t)$. The time dependence of these operatores is analytical since it results from the time evolution for the time interval T_z given by the Schrödinger equation on the primary level. Note that it is understood that all the switching instants are chosen relative to T_z. Then a UDD sequence of duration $T_\perp$ can be applied on the secondary level[22] which consists of $N_\perp$ rotations about the spin x or y axis to suppress dephasing up to corrections $\mathcal{O}\left((\gamma T_\perp)^{N_\perp+1}\right)$. This means that general decoherence can be suppressed by a RUDD on the primary and a UDD on the secondary level.

To obtain a quadratic scheme using pulses of finite duration we use a RUDD also on the secondary level, calling the resulting scheme QRUDD. This is possible since in the derivation of the RUDD for pure dephasing in Sect. III we mentioned that the initial Hamiltonian may display an analytic time dependence. This effective time dependence results here from the pulse sequence on the primary level. Thus it is by construction not present *during* the secondary pulses. These secondary pulses have to be constructed in the presence of general decoherence $\vec{\sigma}\cdot\vec{A}$ so that the explicit solution in Ref. 34 cannot be used. But the equations to be solved are given in sufficient generality in this reference. For a proof-of-existence we refer to the work by Khodjasteh *et al.* where concatenated solutions for such pulses are constructed recursively[35].

Hence, the known mathematical properties of UDD sequences and of π pulses suffice to conclude that even general decoherence can be efficiently suppressed by dynamic decoupling with bounded control by means of this QRUDD scheme. It is a quadratic scheme of UDD sequences of bounded, and thus essentially realistic, control pulses.

VIII. SUMMARY

We derived in this paper that an optimized sequence of realistic pulses (RUDD), i.e., of finite duration and amplitude, can be set up which suppresses dephasing or

longitudinal relaxation up to T^{N+1} in the length of the sequence and up to $\tau_{\rm mx}^M$ in the maximum duration of the pulses, avoiding all mixed terms in contrast to previous proposals.

This statement is based on rigorous analytical calculations for bounded baths and it is expected to apply to systems with hard high-energy cutoff. Our argument is based on the fundamental mathematical property of the optimized sequences of UDD-type, namely a certain antiperiodicity in the auxiliary variable ϑ. Thus it only applies to such sequences and not to arbitrary sequences.

We introduced and exploited the concept of double scaling in the durations $\tau_{\rm p}$ of the pulses *and* in the duration T of the whole sequence. We emphasize that both scales can be varied independently except for a certain constraint, see Eq. (2).

The key achievement is to establish a precise prescription how the sequence has to be adjusted to allow for the use of pulses with bounded amplitudes, which are thus decisively more realistic. Only the adjustment of the sequence to the use of tailored pulses of finite duration allowed us to circumvent no-go theorems[25,26] concerning the properties of tailored pulses.

The proposed RUDD can be used for suppressing pure dephasing, i.e., suppressing coupling of the bath to one spin component, or for suppressing longitudinal relaxation, i.e., suppressing coupling of the bath to two spin components. General decoherence, i.e., suppressing coupling of the bath to all three spin components, cannot be suppressed by a single RUDD but by a quadratic concatenated scheme (QRUDD) of two RUDDs, made from rotations about two perpendicular spin axes.

Based on the known properties of UDD[12,13,15] and QDD[21,22], we think that the design of the sequences is very close to its optimum. But we expect that the design of the pulses can still be improved. While for $M = 3$ (leading non-vanishing correction is cubic in $\tau_{\rm p}$) rather simple pulse shapes are known[34], for higher order pulses with $M > 3$ recursive concatenation provides a recipe for their construction at the expense of an exponential increase in the number of elementary pulses[35].

Certainly, further research is called for to determine the performance of RUDD and QRUDD for specific models. One important issue is to determine the size of the prefactors of the neglected terms. Another issue on the way to the experimental application of RUDD and QRUDD is to investigate the robustness of both the tailored pulses and the sequences to imperfections such as imprecise timing.

To stimulate further research on the experimental side we proposed an experimental setup to verify the RUDD for simulated noise which can be switched off[16–18] so that a RUDD can be checked without pulse shaping.

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